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13. ABSTRACT (Maximum 200 words)

The objectives of this project were to extend the Stroh formalism for two-dimensional deformations of anisotropic elastic bodies to include heat flow and piezoelectricity, and to apply the theory to study the effects of temperature and piezoelectricity in composites. This report summarizes the accomplishment of the investigation. Several basic solutions are obtained that are fundamental in analyzing more realistic problems in composites. For instance, the displacement at an interface crack between two materials in a composite is oscillatory when the two materials are "mismatched". This leads to the physically unacceptable interpenetration of the crack surfaces. However it is saved from a higher order stress singularity when the effects of heat flow is included. On the other hand, when the two materials are "not mismatched", the interpenetration does not occur but a higher order stress singularity may arise due to heat flow.

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Anisotropic Elastic Materials and Composites With The Effects of Temperature and Piezoelectricity

The objectives of this project were to extend the Stroh formalism for two-dimensional deformations of anisotropic elastic bodies to include heat flow and piezoelectricity, and to apply the theory to study the effects of temperature and piezoelectricity in composites. The basic Stroh formalism is not new. The foundation of the formalism was laid down by Stroh in his pioneering papers in 1958 and 1962. Since then material scientists, physicists, and applied mathematicians have recognized its superior features and have further developed the formalism. For unexplainable reasons the engineering community has refused to employ Stroh formalism and have steadfastly employed the traditional Lekhnitskii formalism. Only in recent years did a few young researchers in the applied mechanics community begin to employ the Stroh formalism. The Stroh formalism is mathematically elegant and technically powerful. It is not difficult to understand. In fact it is easier to learn and simpler to use than the Lekhnitskii formalism. In view of this, it is even more puzzling why the engineering community has been so slow in catching up with the Stroh formalism. Researchers in material sciences, physics, and applied mathematics hardly use the Lekhnitskii formalism.

In the first year of the project we extended the Stroh formalism to include heat flow in general anisotropic elastic materials [1]. We then studied the stress singularities at an interface crack due to heat flow in a bimaterial that consists of two dissimilar general anisotropic materials. If the stress singularities at the crack tip are written in the form of r^{δ}, there are three stress singularities

$$r^{-\frac{1}{2}}, \quad r^{-\frac{1}{2}+i\gamma}, \quad \text{and} \quad r^{-\frac{1}{2}-i\gamma} \tag{1}$$

when the heat effects are ignored. The materials in the composite are mismatched when $\gamma \neq 0$ so that the second and the third δ in (1) are complex. With the inclusion of heat flow, it is shown that there is a possibility of a fourth stronger singularity

$$r^{-\frac{1}{2}}(\ln r). \tag{2}$$

This is a stronger singularity than the ones shown in (1), indicating that heat flow increases the potential for initiation of fracture in composites. The conditions for the existence of the stronger singularity (2) are presented explicitly.

The conditions for the existence of the stronger singularity derived in [1] are applicable to general anisotropic materials. It is difficult to interpret the conditions physically. In order to have a quantitative result, monoclinic materials with the symmetry planes at x_3=0 are considered [2]. All quantities involved in the solutions and in the conditions for the existence of the stronger singularity now have an explicit real form expression. An unexpected result is that the stronger singularity does not exist if the two materials in the bimaterial are mismatched. The is a rather puzzling result. One would have expected that the reverse is true. It is known that a mismatched bimaterial induces an oscillatory displacement

near the crack tip, resulting in the physically unacceptable interpenetration of the two materials at the interface crack. It is, however, saved from a stronger stress singularity when the effects of heat flow is included. On the other hand, while a non-mismatched bimaterial is immune to the interpenetration of the two materials at the interface crack, it may admit a stronger stress singularity when the heat flow in considered.

For isotropic bimaterials the two materials are not mismatched if

$$\frac{1-2\nu}{\mu}=\frac{1-2\nu'}{\mu'} \tag{3}$$

where μ is the shear modulus and ν is the Poisson's ratio. The prime refers to the second material. The condition for existence of the stronger singularity is

$$\frac{\alpha(1+\nu)}{\kappa}\neq\frac{\alpha'(1+\nu')}{\kappa'} \tag{4}$$

where α and κ are the thermal expansion coefficient and the heat conductivity, respectively. Thus when (3) and (4) are satisfied, there exists a stronger stress singularity of the form (2) in an isotropic bimaterial. If the material is incompressible, i.e., if $\nu=1/2$, (3) is automatically satisfied and (4) leads to

$$\alpha\kappa'\neq\alpha'\kappa. \tag{5}$$

In the second and third year we looked at the extension of the Stroh formalism to piezoelectric materials and its applications. The sextic formalism of Stroh becomes an octet formalism. Most of the identities for anisotropic materials can be extended to piezoelectric materials except that certain matrices are no longer positive definite. However, they can be shown to be nonsingular, a very important and crucial property in the analysis of piezoelectric materials [3].

One of the basic problems is that of the Green's function for the infinite space subject to a prescribed singularity at the origin of the coordinate system. The usefulness of Green's function in solving practical problems with an arbitrarily prescribed boundary conditions on an irregular shape of boundary is well recognized. We presented in [3] the Green's function in the infinite space of homogeneous piezoelectric material subjected to a line force, line charge, and a line dislocation. To make it more general, we consider the infinite space to consist of an arbitrary number of wedges of different wedge angles and different piezoelectric materials. The Green's function due to a line force, a line charge, and a line dislocation at the center of this composite space consisting of an arbitrary number of wedges, has a surprisingly simple expression. Moreover, the solution can be expressed in a real form, not in a complex form. A real and closed form solution for the Green's function is very useful in applications.

A by-product of the paper in [3] is the problem of a composite wedge subjected to a line force at the apex of the composite wedge. The composite wedge now consists of any number of homogeneous piezoelectric wedges of different wedge angles. Again, the solution can be expressed in a real and closed form.

When the number of the wedges in the composite space or composite wedge becomes infinite, the material is an angularly inhomogeneous material. This means that the material property depends on the polar angle θ in a cylindrical coordinate system (r, θ, z). The Stroh formalism does not apply to an inhomogeneous materials, let alone the special case of angularly inhomogeneous materials. A different approach must be employed. This is presented in [4]. The solution is again in an explicit real form. Real materials may not always be homogeneous. Glass fibers, for instance, can

be *cylindrical anisotropic*. This is a special case of *angularly inhomogeneous* anisotropic materials.

The papers [3,4] inspire the Russian scientist, Professor V. I. Alshits and the French scientist, Professor H. O. K. Kirchner. They pointed out that, from a different angle, the results in [3,4] can be re-derived by a simpler approach. In fact the alternate derivation also allows one to include piezomagnetic and magnetoelectric properties. This results in a joint paper [5]. The contribution of the principal investigator on this paper is the derivation of governing differential equations employing a *dual coordinate* system. The dependent variables are referred to a rectangular coordinate system while the independent variables are refereed to a cylindrical coordinate system. The employment of a *dual coordinate* system follows from a general dual coordinate system presented in a book by the principal investigator on "Anisotropic Elasticity: Theory and Applications" currently under printing by Oxford University Press.

The last area of applications under this project is the problem of a defect in a material. A defect may have the form of a void or an inclusion of different material. We consider the void to have the shape of an ellipse, and the inclusion to be an rigid elliptic inclusion or an inclusion of dissimilar material [6]. A crack is a special case of an elliptic hole when the minor axis of the ellipse becomes zero. Various problems are studied. For the case of an elliptic hole, explicit real form solutions are obtained for the stress along the elliptic hole boundary subjected to an arbitrary prescribed boundary condition at the elliptic hole boundary. For an elliptic rigid inclusion subjected to a line force, a torque, and a line charge, a real form solution at the interface between the piezoelectric material and the rigid inclusion is obtained. Also obtained is the general solution for an piezo-

electric inclusion of dissimilar materials under a uniform loading at infinity. The stresses along the interface between the two materials are obtained explicitly. In particular, the stress concentration along the interface can be deduced directly from the solutions. These are useful information in applications in which one is interested in the stress localization in the material due to a defect, and in the possible debonding of a fiber in a composite.

In summary, we have generalized the Stroh formalism for anisotropic elastic materials to include the effects of heat flow and piezoelectric property. The generalized formalism allows us to extend many solutions for purely anisotropic materials to materials subjected to heat flow or has piezoelectric effects. Important results are obtained for Green's functions for the infinite space, composite space, and the infinite space with an elliptic hole or a rigid inclusion. Equally important are the stress concentration around an elliptic hole, elliptic rigid inclusion, or an elliptic interface between two dissimilar piezoelectric materials subjected to various loading on the materials. These are very useful in applications, especially in composite materials.

Personnel

Two graduate students, Gongpu Yan and Y. M. Chung were supported by the project. They received their Ph. D. in 1994 and 1995, respectively. Their theses are published in [1]–[5].

Publications under this project

[1] Ting, T. C. T., and Yan, Gongpu (1992) "The $r^{-1/2}(\ln r)$ singularity at interface cracks in anisotropic bimaterials due to heat flow," *J. Thermal Stresses*, **15**, 85-99.

[2] Yan, Gongpu, and Ting, T. C. T. (1993) "The $r^{-1/2}(\ln r)$ singularities at interface cracks in monoclinic and isotropic bimaterials due to heat flow," *J. Appl. Mech.* **60**, 432-437.

[3] Chung, Y. M., and Ting, T. C. T. (1995) "Line force, charge, and dislocation in anisotropic piezoelectric composite wedges and spaces," *J. Appl. Mech.* **620**, 423-428.

[4] Chung, Y. M., and Ting, T. C. T. (1995) "Line force, charge and dislocation in angularly inhomogeneous anisotropic piezoelectric wedges and spaces," *Phil Mag. A*, **71**, 1335-1343.

[5] Alshits, V. I., Kirchner, H. O. K., and Ting, T. C. T. (1995) "Angularly inhomogeneous piezoelectric piezomagnetic magnetoelectric anisotropic media," *Phil. Mag. Letters* **71**, 285-288.

[6] Chung, M. Y., and Ting, T. C. T. (1996) "Piezoelectric solid with an elliptic inclusion or hole," *Int. J. Solids Structures* in press.

Publications [1]–[5] are available in the literature. Paper [6] is not yet published, and is attached in the following pages.

Piezoelectric Solid with an Elliptic Inclusion or Hole

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Abstract: The two-dimensional problem of an elliptic hole in a solid of anisotropic piezoelectric material is studied. The Stroh formalism is adopted here. Real form solutions are obtained along the hole boundary in the case of an arbitrarily prescribed vector field on the hole surface. For an elliptic rigid inclusion of electric conductor subjected to a line force, a torque, and a line charge, a real form solution at the interface is obtained. Finally, general solutions for an elliptic piezoelectric inclusion with uniform loading at infinity are investigated.

1. Introduction

In 1958 and 1962, Stroh elaborated the work of Eshelby *et al.* (1953) on two-dimensional problems of general anisotropic elasticity involving dislocations, line forces, and steady waves. This powerful and elegant approach was named the Stroh formalism.

In 1975, Barnett and Lothe extended Stroh's 1962 paper to include the piezoelectric effect in which an eight-dimensional framework had been developed. Here, we consider the two-dimensional problem of an elliptic hole in a solid of anisotropic piezoelectric material. Similar problems had been studied by Pak (1992) and Sosa (1991). Although some useful solutions had been derived in these two papers, they were both restricted to the transversely isotropic situation. In Pak's 1992 paper, special remote loading conditions were employed and the concentration effect was studied. Likewise, only remote loadings were considered in Sosa's 1991 paper.

Here, solutions of an arbitrarily prescribed loading on the hole surface are derived.

Furthermore, in the case of an elliptic rigid inclusion of electric conductor subjected to a line force, a torque, and a free line charge, real form solutions along the elliptic interface are obtained which could be used to examine the concentration effect. Finally, we investigate the situation of an elliptic piezoelectric inclusion with uniform loading at infinity.

In the following basic solutions of the Stroh formalism with the piezoelectric effect are given. Some boundary conditions are shown in section 2. In sections 3 and 4, a few useful relations are derived. General field solutions to the elliptic problem are obtained in section 5 with emphasize on solutions along the elliptic boundary. Such boundary solutions could be employed to investigate the concentration effect. However, arbitrary constant vectors are involved and remain unknown. They will be determined in sections 6, 7, and 8 in which different boundary conditions are applied.

In a Cartesian coordinate system (x_1, x_2, x_3) the constitutive equations for piezoelectric materials are given by (Tiersten, 1969)

$$\sigma_{ij} = C_{ijkm} u_{k,m} + e_{mij} \varphi_{,m}, \quad D_i = e_{ikm} u_{k,m} - \omega_{im} \varphi_{,m} \quad (i,j,k,m = 1,2,3) \tag{1.1}$$

in which repeated indices mean summation and a comma stands for partial differentiation. σ_{ij} is the elastic stress and D_i is the electric displacement. Coefficients C_{ijkm}, e_{mij}, ω_{im} are, respectively, the elastic stiffnesses, piezoelectric constants, and permittivities with the following symmetries:

$$C_{ijkm} = C_{jikm} = C_{kmij}, \quad e_{mij} = e_{mji}, \quad \omega_{im} = \omega_{mi}. \tag{1.2}$$

u_k is the elastic displacement and φ is the electrostatic potential. C_{ijkm} and ω_{im} are positive definite in the sense that

$$C_{ijkm} u_{i,j} u_{k,m} > 0, \quad \omega_{im} E_i E_m > 0 \tag{1.3}$$

for arbitrary real nonzero $u_{i,j}$ and E_i with

$$E_i = -\varphi_{,i}. \tag{1.4}$$

In the absence of body forces and free charges, the balance laws require

$$\sigma_{ij,j} = 0, \quad D_{i,i} = 0. \tag{1.5}$$

For two-dimensional deformations in which u_k and φ depend on x_1 and x_2 only, a general solution to (1.5) is given by

$$u_J = a_J f(z) \quad (J = 1, 2, 3, 4) \tag{1.6}$$

in which

$$z = x_1 + px_2, \quad u_4 = \varphi, \tag{1.7}$$

and p, a_J are constants to be determined. In matrix notation,

$$\mathbf{u} = \mathbf{a}f(z). \tag{1.8}$$

Thus $\mathbf{u}$, $\mathbf{a}$ are four-vectors and $\mathbf{u}$ is called the generalized displacement. By defining

$$\mathbf{Q} = \begin{bmatrix} \mathbf{Q}^{\mathrm{E}} & \mathbf{e}_{11} \\ \mathbf{e}_{11}^{\mathrm{T}} & -\omega_{11} \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} \mathbf{R}^{\mathrm{E}} & \mathbf{e}_{21} \\ \mathbf{e}_{12}^{\mathrm{T}} & -\omega_{12} \end{bmatrix}, \quad \mathbf{T} = \begin{bmatrix} \mathbf{T}^{\mathrm{E}} & \mathbf{e}_{22} \\ \mathbf{e}_{22}^{\mathrm{T}} & -\omega_{22} \end{bmatrix}, \tag{1.9}$$

where

$$(\mathbf{Q}^{\mathrm{E}})_{ik} = C_{i1k1}, \quad (\mathbf{R}^{\mathrm{E}})_{ik} = C_{i1k2}, \quad (\mathbf{T}^{\mathrm{E}})_{ik} = C_{i2k2}, \quad (\mathbf{e}_{ij})_m = e_{ijm}, \tag{1.10}$$

we combine (1.1), (1.5), and (1.6) into one equation as

$$\left[\mathbf{Q} + p(\mathbf{R} + \mathbf{R}^{\mathrm{T}}) + p^2\mathbf{T}\right]\mathbf{a} = \mathbf{0}. \tag{1.11}$$

The 4×4 matrices $\mathbf{Q}$ and $\mathbf{T}$ are symmetric but not positive definite. However, they can be shown to be nonsingular.

Let the generalized stress function vector $\boldsymbol{\phi}$ be defined as

$$\boldsymbol{\phi} = \mathbf{b}f(z), \quad \mathbf{b} = (\mathbf{R}^{\mathrm{T}} + p\mathbf{T})\mathbf{a} = \frac{-1}{p}(\mathbf{Q} + p\mathbf{R})\mathbf{a}, \tag{1.12}$$

with

$$\sigma_{i1} = -\phi_{i,2}, \quad \sigma_{i2} = \phi_{i,1}, \quad D_1 = -\phi_{4,2}, \quad D_2 = \phi_{4,1}. \tag{1.13}$$

The second equality in $(1.12)_2$ follows from (1.11). Equation (1.13) provide all components of σ_{ij} and D_i except σ_{33} and D_3; they can be determined from (1.1).

With the positive definiteness of C_{ijkm} and ω_{im} shown in (1.3), the eigenvalues p of (1.11) are all complex and consist of four pairs of complex conjugates. Let

$$p_{\alpha+4} = \overline{p_\alpha}, \quad Im\{p_\alpha\} > 0 \quad (\alpha = 1,2,3,4), \tag{1.14}$$

$$\mathbf{a}_{\alpha+4} = \overline{\mathbf{a}_\alpha}, \quad \mathbf{b}_{\alpha+4} = \overline{\mathbf{b}_\alpha}, \tag{1.15}$$

where the overbar denote the complex conjugates. The general solution obtained by superposing eight solutions of (1.8) and $(1.12)_1$ associated with the eight eigenvalues p_α are

$$\mathbf{u} = 2Re\left\{\sum_{\alpha=1}^{4} \mathbf{a}_\alpha f_\alpha(z_\alpha)\right\}, \quad \boldsymbol{\phi} = 2Re\left\{\sum_{\alpha=1}^{4} \mathbf{b}_\alpha f_\alpha(z_\alpha)\right\} \tag{1.16}$$

in which Re stands for the real part and $f_{\alpha+4} = \overline{f_\alpha}$ $(\alpha = 1,2,3,4)$ is chosen.

In most applications

$$f_\alpha(z_\alpha) = q_\alpha f(z_\alpha) \quad (\alpha \text{ not summed}) \tag{1.17}$$

is assumed. Hence, equation (1.16) reduces to, in matrix notation,

$$\mathbf{u} = 2Re\{\mathbf{A} < f(z_*) > \mathbf{q}\}, \quad \boldsymbol{\phi} = 2Re\{\mathbf{B} < f(z_*) > \mathbf{q}\} \tag{1.18}$$

where $\mathbf{A}$ and $\mathbf{B}$ are 4×4 matrices given by

$$\mathbf{A} = [\mathbf{a}_1, \ \mathbf{a}_2, \ \mathbf{a}_3, \ \mathbf{a}_4], \quad \mathbf{B} = [\mathbf{b}_1, \ \mathbf{b}_2, \ \mathbf{b}_3, \ \mathbf{b}_4], \tag{1.19}$$

and $<f(z_*)>$ is the 4×4 diagonal matrix

$$<f(z_*)> = diag< f(z_1), \ f(z_2), \ f(z_3), \ f(z_4) >. \tag{1.20}$$

The elements of the four-vector $\mathbf{q}$ are q_α $(\alpha = 1,2,3,4)$. Notice that the solutions given in (1.18) are in terms of the arbitrary function $f(z_\alpha)$ and the arbitrary complex constant vector $\mathbf{q}$.

2. Boundary Conditions

Consider an arc or a contour $\mathbf{C}$ described by

$$\mathbf{C}(s) : \begin{cases} x_1 = x_1(s) \\ x_2 = x_2(s) \end{cases}, \quad \left[x_1'(s)\right]^2 + \left[x_2'(s)\right]^2 = 1, \tag{2.1}$$

where s is the arc-length. The unit tangential vector $\mathbf{n}$ and the unit normal vector $\mathbf{m}$ are given by

$$\mathbf{n}^{\mathrm{T}} = \left[\frac{dx_1}{ds}, \ \frac{dx_2}{ds}, \ 0\right], \quad \mathbf{m}^{\mathrm{T}} = \left[-\frac{dx_2}{ds}, \ \frac{dx_1}{ds}, \ 0\right], \tag{2.2}$$

respectively. By taking derivative of $\boldsymbol{\phi}$ in the direction of increasing s (with material on the RIGHT-hand side) and using (1.13), we obtain

$$\frac{d\phi_j}{ds} = t_j \quad (j = 1,2,3), \qquad \frac{d\phi_4}{ds} = \mathbf{D}\cdot\mathbf{m} = D_{\mathrm{m}}, \tag{2.3}$$

in which t_j is the component of surface traction vector. Similarly, one obtains

$$\frac{du_4}{ds} = -\mathbf{E}\cdot\mathbf{n} = -E_{\mathrm{n}}. \tag{2.4}$$

If we consider a dielectric interface with materials indicated by "1" and "2", the electrical conditions at the interface are

$$\mathbf{E}_1\cdot\mathbf{n} = \mathbf{E}_2\cdot\mathbf{n}, \quad \mathbf{D}_1\cdot\mathbf{m}_1 + \mathbf{D}_2\cdot\mathbf{m}_2 = \sigma_s \tag{2.5}$$

where $\mathbf{n}$ is a unit vector tangential to the dielectric interface, $\mathbf{m}_{\mathrm{i}}$ is an inward normal unit vector, and σ_s is the free surface charge density along the interface. Without loss in generality, we can rewrite (2.5) as

$$\varphi_1 = \varphi_2, \quad \mathbf{D}_1\cdot\mathbf{m}_1 + \mathbf{D}_2\cdot\mathbf{m}_2 = \sigma_s. \tag{2.6}$$

If we have an interface between electric conductor "1" and dielectric "2", then inside the electric conductor,

$$\mathbf{D}_1 = \mathbf{0}, \quad \mathbf{E}_1 = \mathbf{0}. \tag{2.7}$$

In the dielectric, at the interface

$$\mathbf{E}_2 \cdot \mathbf{n} = E_{2\mathrm{n}} = 0, \quad \mathbf{D}_2 \cdot \mathbf{m}_2 = \sigma_s. \tag{2.8}$$

3. Eight-Dimensional Formalism

The two equations in $(1.12)_2$ can be rewritten as

$$\begin{bmatrix} -\mathbf{R}^{\mathrm{T}} & \mathbf{I} \\ -\mathbf{Q} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix} = p \begin{bmatrix} \mathbf{T} & \mathbf{0} \\ \mathbf{R} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix}. \tag{3.1}$$

Since $\mathbf{T}^{-1}$ exists, we can reduce (3.1) to

$$\mathbf{N}\boldsymbol{\xi} = p\boldsymbol{\xi}, \tag{3.2}$$

where

$$\mathbf{N} = \begin{bmatrix} \mathbf{N}_1 & \mathbf{N}_2 \\ \mathbf{N}_3 & \mathbf{N}_1^{\mathrm{T}} \end{bmatrix}, \quad \boldsymbol{\xi} = \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix}, \tag{3.3}$$

$$\mathbf{N}_1 = -\mathbf{T}^{-1}\mathbf{R}^{\mathrm{T}}, \quad \mathbf{N}_2 = \mathbf{T}^{-1}, \quad \mathbf{N}_3 = \mathbf{R}\mathbf{T}^{-1}\mathbf{R}^{\mathrm{T}} - \mathbf{Q}. \tag{3.4}$$

The real 4×4 matrices $\mathbf{N}_2$ and $\mathbf{N}_3$ are symmetric. Equation (3.2) is a standard eigenrelation in the eight-dimensional space. There are eight eigenvalues p_α $(\alpha = 1, 2, ..., 8)$ and eight associated eigenvectors $\boldsymbol{\xi}_\alpha$. The eigenvalues are the roots of the determinant

$$\|\mathbf{N} - p\mathbf{I}\| = 0. \tag{3.5}$$

The vector $\boldsymbol{\xi}$ in (3.2) is a right eigenvector. The left eigenvector $\boldsymbol{\eta}$ is defined by

$$\boldsymbol{\eta}^{\mathrm{T}}\mathbf{N} = p\boldsymbol{\eta}^{\mathrm{T}}, \quad \mathbf{N}^{\mathrm{T}}\boldsymbol{\eta} = p\boldsymbol{\eta}, \tag{3.6}$$

and can be shown to be

$$\boldsymbol{\eta} = \begin{bmatrix} \mathbf{b} \\ \mathbf{a} \end{bmatrix}. \tag{3.7}$$

Normalization of $\boldsymbol{\xi}_\alpha$ and $\boldsymbol{\eta}_\beta$ (which are orthogonal to each other) gives

$$\boldsymbol{\eta}_\beta^T \boldsymbol{\xi}_\alpha = \delta_{\beta\alpha} \tag{3.8}$$

where $\delta_{\beta\alpha}$ is the Kronecker delta. Making use of (1.15), (1.19), $(3.3)_2$, and (3.7), equation (3.8) is written as

$$\begin{bmatrix} \mathbf{B}^T & \mathbf{A}^T \\ \bar{\mathbf{B}}^T & \bar{\mathbf{A}}^T \end{bmatrix} \begin{bmatrix} \mathbf{A} & \bar{\mathbf{A}} \\ \mathbf{B} & \bar{\mathbf{B}} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}. \tag{3.9}$$

This is the orthogonality relation. The two 8×8 matrices on the left hand side of (3.9) are the inverses of each other. Their product commutes so that

$$\begin{bmatrix} \mathbf{A} & \bar{\mathbf{A}} \\ \mathbf{B} & \bar{\mathbf{B}} \end{bmatrix} \begin{bmatrix} \mathbf{B}^T & \mathbf{A}^T \\ \bar{\mathbf{B}}^T & \bar{\mathbf{A}}^T \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}. \tag{3.10}$$

This is the closure relation and is equivalent to

$$\mathbf{AB}^T + \bar{\mathbf{A}}\bar{\mathbf{B}}^T = \mathbf{I} = \mathbf{BA}^T + \bar{\mathbf{B}}\bar{\mathbf{A}}^T, \quad \mathbf{AA}^T + \bar{\mathbf{A}}\bar{\mathbf{A}}^T = \mathbf{0} = \mathbf{BB}^T + \bar{\mathbf{B}}\bar{\mathbf{B}}^T. \tag{3.11}$$

Hence, the three matrices $\mathbf{S}, \mathbf{H}, \mathbf{L}$ defined by

$$\mathbf{S} = i(2\mathbf{AB}^T - \mathbf{I}), \quad \mathbf{H} = i2\mathbf{AA}^T, \quad \mathbf{L} = -i2\mathbf{BB}^T \tag{3.12}$$

are real. The matrices $\mathbf{H}$ and $\mathbf{L}$ are symmetric and nonsingular (Lothe and Barnett, 1976).

Since $\mathbf{S}, \mathbf{H}, \mathbf{L}$ are real, the following relation exists (Chung, 1995; Ting and Yan, 1991)

$$\begin{bmatrix} \mathbf{S} & \mathbf{H} \\ -\mathbf{L} & \mathbf{S}^T \end{bmatrix} \begin{bmatrix} \mathbf{S} & \mathbf{H} \\ -\mathbf{L} & \mathbf{S}^T \end{bmatrix} = - \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \tag{3.13}$$

which indicates that $\mathbf{SH}$ and $\mathbf{LS}$ are anti-symmetric. It can be shown that $\mathbf{H}^{-1}\mathbf{S}$ and $\mathbf{SL}^{-1}$ are also anti-symmetric.

Finally, we rewrite (3.2) as

$$\mathbf{N}\begin{bmatrix}\mathbf{a}f'(z)\\ \mathbf{b}f'(z)\end{bmatrix} = \begin{bmatrix}\mathbf{a}pf'(z)\\ \mathbf{b}pf'(z)\end{bmatrix}. \tag{3.14}$$

Employing $(1.7)_1$, (1.8), and $(1.12)_1$ leads to a matrix differential equation for $\mathbf{u}$ and $\boldsymbol{\phi}$,

$$\mathbf{N}\begin{bmatrix}\mathbf{u}_{,1}\\ \boldsymbol{\phi}_{,1}\end{bmatrix} = \begin{bmatrix}\mathbf{u}_{,2}\\ \boldsymbol{\phi}_{,2}\end{bmatrix}. \tag{3.15}$$

4. The Integral Formalism

Let the tensor E_{iJKm} be defined by (Barnett and Lothe, 1975; Kuo and Barnett, 1991)

$$\begin{aligned} E_{iJKm} &= C_{ijkm} && (J,K = 1,2,3),\\ &= e_{mij} && (J = 1,2,3;\ K = 4),\\ &= e_{ikm} && (J = 4;\ K = 1,2,3),\\ &= -\omega_{im} && (J = K = 4). \end{aligned} \tag{4.1}$$

With $\mathbf{n}(\omega)$ and $\mathbf{m}(\omega)$ given by

$$\mathbf{n}^{\mathrm{T}}(\omega) = [\cos\omega,\ \sin\omega,\ 0], \quad \mathbf{m}^{\mathrm{T}}(\omega) = [-\sin\omega,\ \cos\omega,\ 0] \tag{4.2}$$

in which ω is a real parameter range from 0 to 2π, we let

$$\begin{gathered} Q_{JK}(\omega) = n_i(\omega)\,E_{iJKm}\,n_m(\omega), \quad R_{JK}(\omega) = n_i(\omega)\,E_{iJKm}\,m_m(\omega),\\ T_{JK}(\omega) = m_i(\omega)\,E_{iJKm}\,m_m(\omega), \end{gathered} \tag{4.3}$$

and

$$\begin{gathered} \mathbf{N}_1(\omega) = -\mathbf{T}^{-1}(\omega)\mathbf{R}^{\mathrm{T}}(\omega), \quad \mathbf{N}_2(\omega) = \mathbf{T}^{-1}(\omega),\\ \mathbf{N}_3(\omega) = \mathbf{R}(\omega)\mathbf{T}^{-1}(\omega)\mathbf{R}^{\mathrm{T}}(\omega) - \mathbf{Q}(\omega). \end{gathered} \tag{4.4}$$

Lothe and Barnett (1976) have shown that

$$\mathbf{S} = \frac{1}{\pi}\int_0^{\pi}\mathbf{N}_1(\omega)d\omega, \quad \mathbf{H} = \frac{1}{\pi}\int_0^{\pi}\mathbf{N}_2(\omega)d\omega, \quad -\mathbf{L} = \frac{1}{\pi}\int_0^{\pi}\mathbf{N}_3(\omega)d\omega. \tag{4.5}$$

Equations (4.5) provide an alternate to (3.12) for the Barnett-Lothe tensors $\mathbf{S}$, $\mathbf{H}$, and $\mathbf{L}$.

5. The General Solutions

In this section the general solutions for two-dimensional deformations along an elliptic boundary will be derived (Ting and Yan, 1991). An ellipse Γ given by

$$\Gamma : \begin{cases} x_1 = a\cos\psi \\ x_2 = b\sin\psi \end{cases} \tag{5.1}$$

is shown in Fig.1. Let $\mathbf{n}$ and $\mathbf{m}$ be the unit vectors tangential and normal to the elliptic boundary, respectively and ω be the angle between vector $\mathbf{n}$ and the positive x_1 axis. Hence,

$$\mathbf{n}^{\mathrm{T}} = [\cos\omega,\ \sin\omega,\ 0], \quad \mathbf{m}^{\mathrm{T}} = [-\sin\omega,\ \cos\omega,\ 0] \tag{5.2}$$

which is (4.2). The infinitesimal arc-length ds of the ellipse is given by

$$ds = \rho(\psi)d\psi, \quad \rho(\psi) = \sqrt{a^2\sin^2\psi + b^2\cos^2\psi}. \tag{5.3}$$

From $(2.2)_1$ and (5.1) we see that

$$\cos\omega = \frac{-a}{\rho(\psi)}\sin\psi, \quad \sin\omega = \frac{b}{\rho(\psi)}\cos\psi \tag{5.4}$$

when comparison is made with $(5.2)_1$.

If there is a line force $\mathbf{f}$ and a free line-charge density λ applied at the origin (Fig.1), by employing (2.3), the equilibrium conditions give

$$\left.\begin{aligned} \oint_\Gamma (\mathbf{t}_{\mathrm{m}})_j ds &= \lim_{B\to A} \phi_j(B) - \phi_j(A) = f_j \quad (j = 1,2,3) \\ \oint_\Gamma \mathbf{D}\cdot\mathbf{m}ds &= \lim_{B\to A} \phi_4(B) - \phi_4(A) = -\lambda \end{aligned}\right\} \tag{5.5}$$

in which $\mathbf{t}_{\mathrm{m}}$ and $\mathbf{D}$ are the surface traction and the electric displacement of the medium along the elliptic boundary Γ, respectively. Therefore, we have a jump in $\boldsymbol{\phi}$ across the positive x_1 axis if $\mathbf{f}$ and λ are not equal to zero. Points A and B are in fact the same point on positive x_1 axis except that when one moves from A to B counter-clockwise, the whole ellipse Γ is transversed.

Consider the transformation

$$z_\alpha = c_\alpha \zeta_\alpha + d_\alpha \zeta_\alpha^{-1} \quad (\alpha = 1,2,3,4), \tag{5.6}$$

where c_α and d_α are complex constants and $z_\alpha = x_1 + p_\alpha x_2$. The constants c_α and d_α are chosen such that when $(x_1, x_2) \subset \Gamma$, ζ_α $(\alpha = 1,2,3,4)$ is on a unit circle. That is,

$$\zeta_\alpha\big|_\Gamma = e^{i\psi} = \cos\psi + i\sin\psi \quad (\alpha = 1,2,3,4) \tag{5.7}$$

when $z_\alpha = a\cos\psi + p_\alpha b\sin\psi$ and one obtains

$$c_\alpha = \frac{a - ip_\alpha b}{2}, \quad d_\alpha = \frac{a + ip_\alpha b}{2}. \tag{5.8}$$

Since a, b, and $Im\{p_\alpha\}$ are all positive and non-zero, it can be shown that the branch points ζ_α of the transformation (5.6) are located inside the unit circle in the ζ_α-plane. Hence, the branch points in the (x_1, x_2) plane are located inside the ellipse. In addition, the transformation is one-to-one outside the elliptic hole.

In order to satisfy the jump conditions stated in (5.5) along the positive x_1 axis, the arbitrary function $f(z_\alpha)$ given in (1.18) is chosen to be

$$f(z_\alpha) = \ln\zeta_\alpha, \tag{5.9}$$

with z_α and ζ_α being related by (5.6). Also, by putting (Ting, 1986; 1988a, Hwu and Ting, 1989)

$$\mathbf{q} = \mathbf{A}^{\mathrm{T}}\mathbf{g}_0 + \mathbf{B}^{\mathrm{T}}\mathbf{h}_0 \tag{5.10}$$

in (1.18) where $\mathbf{g}_0$ and $\mathbf{h}_0$ are real constants, we obtain the first basic solution

$$\left.\begin{aligned} \mathbf{u}^{\mathrm{I}} &= 2Re\left\{\mathbf{A}<\ln\zeta_*>\mathbf{A}^{\mathrm{T}}\right\}\mathbf{g}_0 + 2Re\left\{\mathbf{A}<\ln\zeta_*>\mathbf{B}^{\mathrm{T}}\right\}\mathbf{h}_0 \\ \boldsymbol{\phi}^{\mathrm{I}} &= 2Re\left\{\mathbf{B}<\ln\zeta_*>\mathbf{A}^{\mathrm{T}}\right\}\mathbf{g}_0 + 2Re\left\{\mathbf{B}<\ln\zeta_*>\mathbf{B}^{\mathrm{T}}\right\}\mathbf{h}_0 \end{aligned}\right\} \tag{5.11}$$

in which $<\ln\zeta_*>$ is the diagonal matrix of $\ln\zeta_\alpha$ with $\alpha = 1,2,3,4$. Since $\ln\zeta_\alpha$ is a multi-valued function, a cut along $\psi = 0$ is introduced which makes $\mathbf{u}^{\mathrm{I}}$, $\boldsymbol{\phi}^{\mathrm{I}}$ single-valued and allows a discontinuity along the positive x_1 axis. As $z_\alpha \to \infty$, the elastic stresses and the electric displacements obtained from $(5.11)_2$ vanish. This is consistent with the boundary conditions at

infinity.

In order to provide analytical solutions outside the ellipse,

$$f(z_\alpha) = \zeta_\alpha^{-k}, \quad \mathbf{q} = \mathbf{A}^T\mathbf{g}_k + \mathbf{B}^T\mathbf{h}_k, \quad (k = 1,2,\ldots) \tag{5.12}$$

are assumed in (1.18) where $\mathbf{g}_k$, $\mathbf{h}_k$ are real constants. Superimposing the solutions from $k = 1$ to ∞ leads to the second basic solution

$$\left.\begin{aligned} \mathbf{u}^{II} &= 2\sum_{k=1}^{\infty} Re\left\{\mathbf{A} < \zeta_*^{-k} > \mathbf{A}^T\right\}\mathbf{g}_k + 2\sum_{k=1}^{\infty} Re\left\{\mathbf{A} < \zeta_*^{-k} > \mathbf{B}^T\right\}\mathbf{h}_k \\ \boldsymbol{\phi}^{II} &= 2\sum_{k=1}^{\infty} Re\left\{\mathbf{B} < \zeta_*^{-k} > \mathbf{A}^T\right\}\mathbf{g}_k + 2\sum_{k=1}^{\infty} Re\left\{\mathbf{B} < \zeta_*^{-k} > \mathbf{B}^T\right\}\mathbf{h}_k \end{aligned}\right\} \tag{5.13}$$

in which $< \zeta_*^{-k} >$ is the diagonal matrix of ζ_α^{-k} $(\alpha = 1,2,3,4)$. Notice that both $\mathbf{u}^{II}$ and $\boldsymbol{\phi}^{II}$ approach zero as $z_\alpha \to \infty$ (or $\zeta_\alpha \to \infty$).

With (5.7) it is easy to see that

$$< \ln\zeta_*|_\Gamma > = i\psi\mathbf{I}, \quad < \zeta_*^{-k}|_\Gamma > = \cos(k\psi)\mathbf{I} - i\sin(k\psi)\mathbf{I}. \tag{5.14}$$

Substituting back in (5.11) gives the first basic solution along the elliptic boundary Γ as

$$\mathbf{u}^I|_\Gamma = \psi\hat{\mathbf{h}}_0, \quad \boldsymbol{\phi}^I|_\Gamma = \psi\hat{\mathbf{g}}_0, \tag{5.15}$$

$$\hat{\mathbf{h}}_0 = \mathbf{H}\mathbf{g}_0 + \mathbf{S}\mathbf{h}_0, \quad \hat{\mathbf{g}}_0 = \mathbf{S}^T\mathbf{g}_0 - \mathbf{L}\mathbf{h}_0, \tag{5.16}$$

when using (3.12). Similarly, the second basic solution along the elliptic boundary Γ is in the form

$$\mathbf{u}^{II}|_\Gamma = \sum_{k=1}^{\infty}\left[\cos(k\psi)\mathbf{h}_k - \sin(k\psi)\hat{\mathbf{h}}_k\right], \quad \boldsymbol{\phi}^{II}|_\Gamma = \sum_{k=1}^{\infty}\left[\cos(k\psi)\mathbf{g}_k - \sin(k\psi)\hat{\mathbf{g}}_k\right] \tag{5.17}$$

in which

$$\hat{\mathbf{h}}_k = \mathbf{H}\mathbf{g}_k + \mathbf{S}\mathbf{h}_k, \quad \hat{\mathbf{g}}_k = \mathbf{S}^T\mathbf{g}_k - \mathbf{L}\mathbf{h}_k. \tag{5.18}$$

Some useful relations between $\mathbf{g}_k$, $\mathbf{h}_k$, $\hat{\mathbf{g}}_k$, and $\hat{\mathbf{h}}_k$ are given below (Chung, 1995; Ting and Yan, 1991).

$$\mathbf{h}_k = \mathbf{L}^{-1}\left(\mathbf{S}^{\mathrm{T}}\mathbf{g}_k - \hat{\mathbf{g}}_k\right), \quad \hat{\mathbf{h}}_k = \mathbf{L}^{-1}\left(\mathbf{g}_k + \mathbf{S}^{\mathrm{T}}\hat{\mathbf{g}}_k\right) \quad (k = 0,1,2,\ldots), \tag{5.19}$$

$$\hat{\mathbf{g}}_k = -\mathbf{H}^{-1}\left(\mathbf{h}_k + \mathbf{S}\hat{\mathbf{h}}_k\right), \quad \mathbf{g}_k = -\mathbf{H}^{-1}\left(\mathbf{S}\mathbf{h}_k - \hat{\mathbf{h}}_k\right) \quad (k = 0,1,2,\ldots). \tag{5.20}$$

In fact, any two of $\mathbf{g}_k$, $\mathbf{h}_k$, $\hat{\mathbf{g}}_k$, and $\hat{\mathbf{h}}_k$ can be written in terms of the others.

In order to investigate the concentration effect, we will derive the generalized stress vector $\hat{\mathbf{t}}_{\mathrm{m}}$ and the generalized hoop stress vector $\hat{\mathbf{t}}_{\mathrm{n}}$ along the elliptic boundary. In Fig.2, if n is the arc-length of Γ measured in the direction of $\mathbf{n}$, then from (2.3) the generalized stress vector $\hat{\mathbf{t}}_{\mathrm{m}}$ is defined as

$$\hat{\mathbf{t}}_{\mathrm{m}}^{\mathrm{T}} = \left[(\mathbf{t}_{\mathrm{m}})_1,\ (\mathbf{t}_{\mathrm{m}})_2,\ (\mathbf{t}_{\mathrm{m}})_3,\ D_{\mathrm{m}}\right] = \boldsymbol{\phi}^{\mathrm{T}}{}_{,n}, \tag{5.21}$$

or, using $(5.3)_1$,

$$\hat{\mathbf{t}}_{\mathrm{m}} = \boldsymbol{\phi}_{,n} = \frac{\partial \boldsymbol{\phi}|_\Gamma}{\rho(\psi)\,\partial\psi}. \tag{5.22}$$

Substituting $\boldsymbol{\phi}^{\mathrm{I}}|_\Gamma$ and $\boldsymbol{\phi}^{\mathrm{II}}|_\Gamma$ given in $(5.15)_2$ and $(5.17)_2$ leads to

$$\hat{\mathbf{t}}_{\mathrm{m}}^{\mathrm{I}} = \frac{1}{\rho(\psi)}\hat{\mathbf{g}}_0, \quad \hat{\mathbf{t}}_{\mathrm{m}}^{\mathrm{II}} = \frac{-1}{\rho(\psi)}\sum_{k=1}^{\infty} k\left[\sin(k\psi)\,\mathbf{g}_k + \cos(k\psi)\,\hat{\mathbf{g}}_k\right]. \tag{5.23}$$

Note that the arbitrary constant vectors $\hat{\mathbf{g}}_0$, $\mathbf{g}_k$, and $\hat{\mathbf{g}}_k$ $(k = 1, 2, \ldots)$ are involved.

Similarly, in Fig.3, if the generalized hoop stress vector $\hat{\mathbf{t}}_{\mathrm{n}}$ is defined by

$$\hat{\mathbf{t}}_{\mathrm{n}}^{\mathrm{T}} = \left[(\mathbf{t}_{\mathrm{n}})_1,\ (\mathbf{t}_{\mathrm{n}})_2,\ (\mathbf{t}_{\mathrm{n}})_3,\ -D_{\mathrm{n}}\right] \tag{5.24}$$

with $-D_{\mathrm{n}} = \mathbf{D} \cdot (-\mathbf{n})$, then by letting m be the arc-length measured in the direction of $\mathbf{m}$, it is clear that

$$\hat{\mathbf{t}}_{\mathrm{n}} = \boldsymbol{\phi}_{,m} = -\boldsymbol{\phi}_{,1}\sin\omega + \boldsymbol{\phi}_{,2}\cos\omega \tag{5.25}$$

where use has been made of (2.3) and $(5.2)_2$. Alternatively, one can express $\hat{\mathbf{t}}_{\mathrm{n}}$ in terms of $\mathbf{u}_{,n}$ and $\hat{\mathbf{t}}_{\mathrm{m}}$ as (Chung, 1995; Ting and Yan, 1991)

$$\hat{\mathbf{t}}_{\mathrm{n}} = \mathbf{N}_3(\omega)\,\mathbf{u}_{,n} + \mathbf{N}_1^{\mathrm{T}}(\omega)\,\hat{\mathbf{t}}_{\mathrm{m}} \tag{5.26}$$

which is a relation applies to a general shape of boundary.

For the elliptic boundary Γ shown in Fig.3, we have

$$\hat{\mathbf{t}}_{\mathrm{n}} = \mathbf{N}_3(\omega)\frac{\partial \mathbf{u}|_{\Gamma}}{\rho(\psi)\partial\psi} + \mathbf{N}_1^{\mathrm{T}}(\omega)\hat{\mathbf{t}}_{\mathrm{m}} = \hat{\mathbf{t}}_{\mathrm{n}}^{\mathrm{I}} + \hat{\mathbf{t}}_{\mathrm{n}}^{\mathrm{II}} \tag{5.27}$$

in which

$$\left.\begin{aligned} \hat{\mathbf{t}}_{\mathrm{n}}^{\mathrm{I}} &= \frac{1}{\rho(\psi)}\left[\mathbf{N}_3(\omega)\hat{\mathbf{h}}_0 + \mathbf{N}_1^{\mathrm{T}}(\omega)\hat{\mathbf{g}}_0\right] \\ \hat{\mathbf{t}}_{\mathrm{n}}^{\mathrm{II}} &= -\frac{\mathbf{N}_3(\omega)}{\rho(\psi)}\sum_{k=1}^{\infty}\left\{k\left[\sin(k\psi)\mathbf{h}_k + \cos(k\psi)\hat{\mathbf{h}}_k\right]\right\} \\ &\quad -\frac{\mathbf{N}_1^{\mathrm{T}}(\omega)}{\rho(\psi)}\sum_{k=1}^{\infty}\left\{k\left[\sin(k\psi)\mathbf{g}_k + \cos(k\psi)\hat{\mathbf{g}}_k\right]\right\} \end{aligned}\right\}, \tag{5.28}$$

with the use of $(5.15)_1$, $(5.17)_1$, and (5.23). Again the arbitrary constant vectors are involved.

In the case of a hole with free surface and electrically open (i.e., zero normal component of electric displacement), (5.26) then takes the form

$$\hat{\mathbf{t}}_{\mathrm{n}} = \mathbf{N}_3(\omega)\mathbf{u}_{,n} \tag{5.29}$$

and $\hat{\mathbf{t}}_{\mathrm{m}} = \mathbf{0}$ (Kuo and Barnett, 1991).

If we have a rigid inclusion of electric conductor inclusion with boundary condition $\mathbf{E}\cdot\mathbf{n} = 0$ given by $(2.8)_1$, it follows from (2.4) that (5.26) is reduced to

$$\hat{\mathbf{t}}_{\mathrm{n}} = \mathbf{N}_3(\omega)\begin{bmatrix}\mathbf{u}_{\mathrm{r},n} \\ \mathbf{0}\end{bmatrix} + \mathbf{N}_1^{\mathrm{T}}(\omega)\hat{\mathbf{t}}_{\mathrm{m}} \tag{5.30}$$

in which $\mathbf{u}_{\mathrm{r}}$ is the rigid body motion of the boundary. Hence,

$$\mathbf{u}_{\mathrm{r}} = \mathbf{u}_0 + \Omega\,\mathbf{e}_3 \times \mathbf{r}_{\Gamma} \tag{5.31}$$

where $\mathbf{u}_0$ is a rigid body translation, Ω is the rotation about x_3 axis and $\mathbf{r}_{\Gamma}$ is the position vector of a point on the boundary. For an elliptic boundary Γ, $\mathbf{r}_{\Gamma} = a\cos\psi\,\mathbf{e}_1 + b\sin\psi\,\mathbf{e}_2$.

Thus,

$$\mathbf{r}_{\Gamma,n} = \mathbf{n}, \quad \mathbf{u}_{\mathrm{r},n} = \Omega\,\mathbf{e}_3 \times \mathbf{r}_{\Gamma,n} = \Omega\,\mathbf{m}. \tag{5.32}$$

Substituting $(5.32)_2$ into (5.30) yields (Chung, 1995)

$$\hat{\mathbf{t}}_{\mathrm{n}} = \mathbf{N}_1^{\mathrm{T}}(\omega)\,\hat{\mathbf{t}}_{\mathrm{m}} \tag{5.33}$$

which is also applicable to circular boundary.

The hoop stress σ_{nn} is given by

$$\sigma_{\mathrm{nn}} = \mathbf{t}_{\mathrm{n}} \cdot (-\mathbf{n}) \tag{5.34}$$

and the two shear stresses are

$$\sigma_{\mathrm{nm}} = \mathbf{t}_{\mathrm{n}} \cdot (-\mathbf{m}), \quad \sigma_{\mathrm{n}3} = \mathbf{t}_{\mathrm{n}} \cdot (-\mathbf{e}_3) \tag{5.35}$$

in which $\mathbf{e}_3^{\mathrm{T}} = [0,\ 0,\ 1]$.

Notice that our solutions are all in terms of arbitrary constant vectors $\mathbf{g}_k$, $\mathbf{h}_k$, $\hat{\mathbf{g}}_k$, $\hat{\mathbf{h}}_k$ with $k = 0, 1, 2, \ldots$. When we determine these constants, we have the solutions. In the next three sections, the arbitrary constant vectors will be determined by applying appropriate boundary conditions.

6. An Elliptic Hole

We consider an elliptic hole shown in Fig.2. Let

$$\left.\begin{aligned}
\mathbf{u}|_{\Gamma} &= \mathbf{u}^{\mathrm{I}}|_{\Gamma} + \mathbf{u}^{\mathrm{II}}|_{\Gamma} = \psi\hat{\mathbf{h}}_0 + \sum_{k=1}^{\infty}\left[\cos(k\psi)\,\mathbf{h}_k - \sin(k\psi)\,\hat{\mathbf{h}}_k\right] \\
\boldsymbol{\phi}|_{\Gamma} &= \boldsymbol{\phi}^{\mathrm{I}}|_{\Gamma} + \boldsymbol{\phi}^{\mathrm{II}}|_{\Gamma} = \psi\hat{\mathbf{g}}_0 + \sum_{k=1}^{\infty}\left[\cos(k\psi)\,\mathbf{g}_k - \sin(k\psi)\,\hat{\mathbf{g}}_k\right]
\end{aligned}\right\}. \tag{6.1}$$

The right hand sides of (6.1) are given by (5.15) and (5.17). Since $\mathbf{u}|_{\Gamma}$ must be single-valued, it follows from $(6.1)_1$ that

$$\hat{\mathbf{h}}_0 = \mathbf{0}, \tag{6.2}$$

and one obtains

$$\mathbf{h}_0 = -\mathbf{S}^{-1}\mathbf{H}\,\mathbf{g}_0 \tag{6.3}$$

when $(5.20)_2$ is employed. The generalized stress vector along the elliptic hole boundary Γ is, with (5.23),

$$\hat{\mathbf{t}}_{\mathrm{m}} = \hat{\mathbf{t}}_{\mathrm{m}}^{\mathrm{I}} + \hat{\mathbf{t}}_{\mathrm{m}}^{\mathrm{II}} = \frac{1}{\rho(\psi)}\hat{\mathbf{g}}_0 - \frac{1}{\rho(\psi)}\sum_{k=1}^{\infty}\left\{k\left[\sin(k\psi)\,\mathbf{g}_k + \cos(k\psi)\,\hat{\mathbf{g}}_k\right]\right\}. \tag{6.4}$$

To find $\hat{\mathbf{t}}_{\mathrm{n}}$, we first substitute (5.19) into $(6.1)_1$. With (6.2) and

$$\mathbf{S}\mathbf{L}^{-1} + \mathbf{L}^{-1}\mathbf{S}^{\mathrm{T}} = \mathbf{0}, \tag{6.5}$$

the generalized displacement vector along Γ becomes

$$\mathbf{u}|_{\Gamma} = -\mathbf{S}\mathbf{L}^{-1}\sum_{k=1}^{\infty}\left[\cos(k\psi)\,\mathbf{g}_k - \sin(k\psi)\,\hat{\mathbf{g}}_k\right] - \mathbf{L}^{-1}\sum_{k=1}^{\infty}\left[\cos(k\psi)\,\hat{\mathbf{g}}_k + \sin(k\psi)\,\mathbf{g}_k\right]. \tag{6.6}$$

Equation (6.5) comes from the anti-symmetric property of $\mathbf{S}\mathbf{L}^{-1}$. From $(5.27)_1$ the generalized hoop stress vector then takes the form

$$\begin{aligned}\hat{\mathbf{t}}_{\mathrm{n}} = \left[\mathbf{N}_1^{\mathrm{T}}(\omega) - \mathbf{N}_3(\omega)\mathbf{S}\mathbf{L}^{-1}\right]\hat{\mathbf{t}}_{\mathrm{m}} + \frac{\mathbf{N}_3(\omega)\mathbf{S}\mathbf{L}^{-1}}{\rho(\psi)}\hat{\mathbf{g}}_0 \\ + \frac{\mathbf{N}_3(\omega)\mathbf{L}^{-1}}{\rho(\psi)}\sum_{k=1}^{\infty}\left\{k\left[\sin(k\psi)\,\hat{\mathbf{g}}_k - \cos(k\psi)\,\mathbf{g}_k\right]\right\}\end{aligned} \tag{6.7}$$

in which $(6.4)_2$ and (6.6) are employed. Alternatively, one obtains

$$\hat{\mathbf{t}}_{\mathrm{n}} = \mathbf{G}_1(\omega)\hat{\mathbf{t}}_{\mathrm{m}} + \frac{\mathbf{G}_3(\omega)}{\rho(\psi)}\left\{\mathbf{S}^{\mathrm{T}}\hat{\mathbf{g}}_0 - \sum_{k=1}^{\infty}\left\{k\left[\sin(k\psi)\,\hat{\mathbf{g}}_k - \cos(k\psi)\,\mathbf{g}_k\right]\right\}\right\} \tag{6.8}$$

with

$$\mathbf{G}_1(\omega) = \mathbf{N}_1^{\mathrm{T}}(\omega) - \mathbf{N}_3(\omega)\mathbf{S}\mathbf{L}^{-1}, \quad \mathbf{G}_3(\omega) = -\mathbf{N}_3(\omega)\mathbf{L}^{-1}. \tag{6.9}$$

Since $\mathbf{N}_3(\omega)$ is symmetric so is $\mathbf{G}_3(\omega)\mathbf{L}$. It can be shown that

$$\mathbf{G}_1(\omega)\mathbf{L} = \mathbf{N}_1^{\mathrm{T}}(\omega)\mathbf{L} - \mathbf{N}_3(\omega)\mathbf{S} \tag{6.10}$$

is also symmetric.

For an arbitrarily prescribed boundary condition along Γ, we define a four-vector $\hat{\boldsymbol{\tau}}_{\mathrm{m}}(\psi)$,

$$\hat{\boldsymbol{\tau}}_{\mathrm{m}}^{\mathrm{T}}(\psi) = \left[(\tau_{\mathrm{m}})_1(\psi),\ (\tau_{\mathrm{m}})_2(\psi),\ (\tau_{\mathrm{m}})_3(\psi),\ \tilde{D}_{\mathrm{m}}(\psi)\right] = \left[\boldsymbol{\tau}_{\mathrm{m}}^{\mathrm{T}}(\psi),\ \tilde{D}_{\mathrm{m}}(\psi)\right] \tag{6.11}$$

in which $(\tau_{\mathrm{m}})_i(\psi)$ $(i = 1, 2, 3)$ are arbitrarily prescribed traction components on Γ while the stress at infinity vanishes. $\tilde{D}_{\mathrm{m}}(\psi)$ $(= \tilde{\mathbf{D}}(\psi)\cdot\mathbf{m})$ is an arbitrarily prescribed normal component of electric displacement of the medium along Γ with the electric displacement vanishes at infinity also. Notice that $\tilde{D}_{\mathrm{m}}(\psi) = 0$ refers to the so-called electrically opened situation (Kuo and Barnett, 1991; Pak, 1992). With the arbitrarily prescribed boundary conditions it is clear that

$$\hat{\boldsymbol{\tau}}_{\mathrm{m}}(\psi) = \hat{\mathbf{t}}_{\mathrm{m}}. \tag{6.12}$$

Employing $(6.4)_2$ and the orthogonality properties between sine and cosine, some of the arbitrary constant vectors in terms of $\hat{\boldsymbol{\tau}}_{\mathrm{m}}(\psi)$ are determined as

$$\left.\begin{aligned}
\hat{\mathbf{g}}_0 &= \frac{1}{2\pi}\int_0^{2\pi} \rho(\psi)\,\hat{\boldsymbol{\tau}}_{\mathrm{m}}(\psi)\,d\psi \\
\mathbf{g}_k &= \frac{-1}{k\pi}\int_0^{2\pi} \rho(\psi)\,\hat{\boldsymbol{\tau}}_{\mathrm{m}}(\psi)\sin(k\psi)\,d\psi \qquad (\,k \ge 1\,) \\
\hat{\mathbf{g}}_k &= \frac{-1}{k\pi}\int_0^{2\pi} \rho(\psi)\,\hat{\boldsymbol{\tau}}_{\mathrm{m}}(\psi)\cos(k\psi)\,d\psi \qquad (\,k \ge 1\,)
\end{aligned}\right\}. \tag{6.13}$$

Equations $(5.19)_2$, (6.2), and $(6.13)_1$ can then be used to find $\mathbf{g}_0$. After that, $\mathbf{h}_k$ and $\hat{\mathbf{h}}_k$ are obtained by employing $(5.19)_1$ and $(5.19)_2$, respectively. In fact $\mathbf{h}_0$ can simply be computed from (6.3).

Indeed, $\hat{\mathbf{g}}_0$ is related to the resultant line force $\mathbf{f}$ applied on Γ and the resultant free line-charge density λ enclosed by Γ. By considering (5.5) and (6.12), the equilibrium equation becomes

$$\int_0^{2\pi} \hat{\boldsymbol{\tau}}_m(\psi)\,\rho(\psi)\,d\psi = \oint_\Gamma \hat{\boldsymbol{\tau}}_m(\psi)\,dn = 2\pi\,\hat{\mathbf{g}}_0 = \begin{bmatrix} \mathbf{f} \\ -\lambda \end{bmatrix} = \hat{\mathbf{f}} \tag{6.14}$$

where n is the arc-length along the elliptic hole boundary Γ.

In the following some special boundary conditions of $\hat{\boldsymbol{\tau}}_m(\psi)$ are considered. We first assume

$$\rho(\psi)\tilde{D}_m(\psi) = \beta = \text{constant}. \tag{6.15}$$

By (6.14), one has

$$\tilde{D}_m(\psi) = \frac{\beta}{\rho(\psi)} = \frac{-\lambda}{2\pi\,\rho(\psi)}. \tag{6.16}$$

Suppose that a uniform pressure $\mathbf{p}$ is applied along the elliptic boundary Γ. i.e.,

$$\boldsymbol{\tau}_m(\psi) = -\mathbf{p}\,\mathbf{m}(\omega). \tag{6.17}$$

With this and $(6.16)_1$, the prescribed generalized stress vector takes the form

$$\hat{\boldsymbol{\tau}}_m^T(\psi) = \left[\mathrm{p}\frac{b}{\rho(\psi)}\cos\psi,\ \mathrm{p}\frac{a}{\rho(\psi)}\sin\psi,\ 0,\ \frac{\beta}{\rho(\psi)} \right] \tag{6.18}$$

in which $(5.2)_2$ and (5.4) are employed. By comparing with $(6.4)_2$ the arbitrary constant vectors $\mathbf{g}_k$ and $\hat{\mathbf{g}}_k$ are easily determined as

$$\hat{\mathbf{g}}_0 = \beta\tilde{\mathbf{e}}_4,\quad \mathbf{g}_1 = -\mathbf{p}a\tilde{\mathbf{e}}_2,\quad \hat{\mathbf{g}}_1 = -\mathbf{p}b\tilde{\mathbf{e}}_1,\quad \mathbf{g}_k = \hat{\mathbf{g}}_k = 0 \quad (\,k \geq 2\,), \tag{6.19}$$

where the four-vectors $\tilde{\mathbf{e}}_I$ are defined as

$$(\tilde{\mathbf{e}}_I)_J = \begin{cases} 1, & I = J \\ 0, & I \neq J \end{cases} \quad (I,J = 1,2,3,4). \tag{6.20}$$

Thus, with (6.6), the generalized displacement vector along the elliptic boundary is simply

$$\mathbf{u}|_\Gamma = \mathbf{S}\mathbf{L}^{-1}\mathrm{p}(x_1\tilde{\mathbf{e}}_2 - x_2\tilde{\mathbf{e}}_1) + \mathbf{L}^{-1}\mathrm{p}(\frac{b}{a}x_1\tilde{\mathbf{e}}_1 + \frac{a}{b}x_2\tilde{\mathbf{e}}_2). \tag{6.21}$$

Similarly, with (6.8), (6.12), and (6.18), the generalized hoop stress vector is given by

$$\hat{\mathbf{t}}_n = \mathbf{G}_1(\omega)\begin{bmatrix} p\sin\omega \\ -p\cos\omega \\ 0 \\ \dfrac{\beta}{\rho(\psi)} \end{bmatrix} + \mathbf{G}_3(\omega)\mathbf{S}^T\begin{bmatrix} 0 \\ 0 \\ 0 \\ \dfrac{\beta}{\rho(\psi)} \end{bmatrix} - \mathbf{G}_3(\omega)p\begin{bmatrix} \dfrac{b}{a}\cos\omega \\ \dfrac{a}{b}\sin\omega \\ 0 \\ 0 \end{bmatrix}. \tag{6.22}$$

Likewise, we can consider a uniform in-plane shear stress τ instead of a uniform pressure $\mathbf{p}$.

In addition to the special boundary condition stated in (6.15), the traction boundary condition described by

$$\rho(\psi)\boldsymbol{\tau}_m(\psi) = \boldsymbol{\gamma} = \text{constant} = \frac{\mathbf{f}}{2\pi} \tag{6.23}$$

is considered. Following the similar procedure given above, we have

$$\mathbf{u}|_\Gamma = \mathbf{0} = \text{constant} \tag{6.24}$$

which implies that the elliptic hole is *not* distorted and the electrostatic potential is *constant* on the hole surface. Consequently, if the elliptic hole is filled with a rigid electric conductor and subjected to a concentrated line force $\mathbf{f}$ and a free line charge density λ at the origin, the generalized stress vector $\hat{\mathbf{t}}_m$ along the interface is simply given by (6.15) and (6.23) (Ting and Yan, 1991). The generalized hoop stress vector is

$$\hat{\mathbf{t}}_n = \frac{1}{\rho(\psi)}\left[\mathbf{G}_1(\omega) + \mathbf{G}_3(\omega)\mathbf{S}^T\right]\begin{bmatrix} \boldsymbol{\gamma} \\ \beta \end{bmatrix} \tag{6.25}$$

which can be shown to be consistent with what is stated in (5.33) when (6.5) and (6.9) are employed.

In the following the solutions for boundary conditions prescribed as

$$\hat{\mathbf{t}}_m(\psi) = \begin{bmatrix} \tilde{\sigma}_{11}m_1 + \tilde{\sigma}_{21}m_2 + \tilde{\sigma}_{31}m_3 \\ \tilde{\sigma}_{12}m_1 + \tilde{\sigma}_{22}m_2 + \tilde{\sigma}_{32}m_3 \\ \tilde{\sigma}_{13}m_1 + \tilde{\sigma}_{23}m_2 + \tilde{\sigma}_{33}m_3 \\ \tilde{D}_1m_1 + \tilde{D}_2m_2 + \tilde{D}_3m_3 \end{bmatrix} = \begin{bmatrix} \tilde{\boldsymbol{\sigma}}^T \\ \tilde{\mathbf{D}}^T \end{bmatrix}\mathbf{m}(\omega) = \begin{bmatrix} \tilde{\boldsymbol{\sigma}} \\ \tilde{\mathbf{D}}^T \end{bmatrix}\mathbf{m}(\omega) \tag{6.26}$$

will be derived. Here, $\tilde{\boldsymbol{\sigma}}$ and $\tilde{\mathbf{D}}$ are the prescribed *uniform* stress field and electric displacement field along the elliptic hole boundary Γ within the medium, respectively. Since $\mathbf{m}^{\mathrm{T}}(\omega) = [-\sin\omega,\ \cos\omega,\ 0]$, we have from (5.4)

$$\hat{\mathbf{t}}_{\mathrm{m}}(\psi) = \frac{-b}{\rho(\psi)}\cos\psi\tilde{\mathbf{t}}_1 - \frac{a}{\rho(\psi)}\sin\psi\tilde{\mathbf{t}}_2, \quad \tilde{\mathbf{t}}_1 = \begin{bmatrix}\tilde{\sigma}_{11}\\ \tilde{\sigma}_{21}\\ \tilde{\sigma}_{31}\\ \tilde{D}_1\end{bmatrix}, \quad \tilde{\mathbf{t}}_2 = \begin{bmatrix}\tilde{\sigma}_{12}\\ \tilde{\sigma}_{22}\\ \tilde{\sigma}_{32}\\ \tilde{D}_2\end{bmatrix}. \tag{6.27}$$

The arbitrary constants are determined by comparing with $(6.4)_2$ which leads to

$$\hat{\mathbf{g}}_0 = \mathbf{0}, \quad \mathbf{g}_1 = a\tilde{\mathbf{t}}_2, \quad \hat{\mathbf{g}}_1 = b\tilde{\mathbf{t}}_1, \quad \mathbf{g}_k = \hat{\mathbf{g}}_k = \mathbf{0} \quad (\ k \geq 2\ ). \tag{6.28}$$

With the use of (5.1) and (6.28), (6.6) reduces to

$$\mathbf{u}|_{\Gamma} = -\mathbf{S}\mathbf{L}^{-1}\left[x_1\tilde{\mathbf{t}}_2 - x_2\tilde{\mathbf{t}}_1\right] - \mathbf{L}^{-1}\left[\frac{b}{a}x_1\tilde{\mathbf{t}}_1 + \frac{a}{b}x_2\tilde{\mathbf{t}}_2\right]. \tag{6.29}$$

The generalized hoop stress vector stated in (6.8) takes the form

$$\hat{\mathbf{t}}_{\mathrm{n}} = \mathbf{G}_1(\omega)\left[\cos\omega\tilde{\mathbf{t}}_2 - \sin\omega\tilde{\mathbf{t}}_1\right] + \mathbf{G}_3(\omega)\left[\frac{b}{a}\cos\omega\tilde{\mathbf{t}}_1 + \frac{a}{b}\sin\omega\tilde{\mathbf{t}}_2\right] \tag{6.30}$$

when (5.4), (6,12), $(6.27)_1$, and (6.28) are employed.

For the problem of an elliptic hole subject to a uniform stress field $\boldsymbol{\sigma}^{\infty}$ and a uniform electric displacement field $\mathbf{D}^{\infty}$ at infinity while the surface of the hole is free of traction and electrically open (Pak, 1992), the solution may be separated into two parts. The first part is the uniform solution in which the stress and electric displacement are $\boldsymbol{\sigma}^{\infty}$ and $\mathbf{D}^{\infty}$ everywhere. The second part is the "disturbed" state due to the presence of the hole. The solution of the second part must satisfy the boundary conditions that the stress and electric displacement vanish at infinity while at the hole surface

$$\hat{\mathbf{t}}_{\mathrm{m}}(\psi) = \begin{bmatrix}-\boldsymbol{\sigma}^{\infty}\\ -\mathbf{D}^{\infty}\end{bmatrix}\mathbf{m}(\omega). \tag{6.31}$$

This is precisely the problem considered in this section.

In general, for an arbitrarily prescribed boundary condition $\hat{\boldsymbol{\tau}}_{\mathrm{m}}(\psi)$ the series solutions $\mathbf{u}|_{\Gamma}$ and $\hat{\mathbf{t}}_{\mathrm{n}}$ given above retain infinite terms. However, by introducing the conjugate function (Bary, 1964; Ting and Yan, 1991), one can rewrite the infinite series solutions in terms of definite integrals.

7. A Rigid Inclusion of Electric Conductor

In section 6 the solutions for a rigid elliptic inclusion of electric conductor in the absence of torque are studied. Here, in addition to a line force $\mathbf{f}$ and a free line-charge density λ, a counter-clockwise torque $T\mathbf{e}_3$ is applied. The generalized stress function vector and generalized displacement vector along the elliptic boundary Γ are, respectively,

$$\boldsymbol{\phi}|_{\Gamma} = \psi\hat{\mathbf{g}}_0 + \sum_{k=1}^{\infty}\left[\cos(k\psi)\,\mathbf{g}_k - \sin(k\psi)\,\hat{\mathbf{g}}_k\right], \quad \mathbf{u}|_{\Gamma} = \sum_{k=1}^{\infty}\left[\cos(k\psi)\,\mathbf{h}_k - \sin(k\psi)\,\hat{\mathbf{h}}_k\right]. \tag{7.1}$$

The equilibrium conditions stated in (5.5) are,

$$-\oint_{\Gamma}\hat{\mathbf{t}}_{\mathrm{m}}dn + \hat{\mathbf{f}} = \mathbf{0}, \quad \hat{\mathbf{f}} = \begin{bmatrix}\mathbf{f}\\ -\lambda\end{bmatrix} \tag{7.2}$$

and $\hat{\mathbf{t}}_{\mathrm{m}}$ is defined in $(5.21)_1$. Using $(5.22)_2$ and $(7.1)_1$, (7.2) reduces to

$$-\int_0^{2\pi}\frac{\partial\boldsymbol{\phi}|_{\Gamma}}{\rho(\psi)\,\partial\psi}\rho(\psi)d\psi + \hat{\mathbf{f}} = \boldsymbol{\phi}(0)|_{\Gamma} - \boldsymbol{\phi}(2\pi)|_{\Gamma} + \hat{\mathbf{f}} = -2\pi\,\hat{\mathbf{g}}_0 + \hat{\mathbf{f}} = \mathbf{0}, \tag{7.3}$$

which is the same result given in (6.14) so that $\hat{\mathbf{g}}_0$ can be computed. Since the rigid inclusion does not deform and the electrostatic potential φ $(= u_4)$ is constant along the elliptic boundary Γ, by ignoring the constant components and noticing that $\mathbf{r}_{\Gamma} = a\cos\psi\,\mathbf{e}_1 + b\sin\psi\,\mathbf{e}_2$, we have, by using (5.31),

$$\mathbf{u}|_{\Gamma} = \Omega\,(a\cos\psi\,\tilde{\mathbf{e}}_2 - b\sin\psi\,\tilde{\mathbf{e}}_1). \tag{7.4}$$

Some of the arbitrary constants are determined by comparing with $(7.1)_2$ which yields

$$\mathbf{h}_1 = \Omega a\tilde{\mathbf{e}}_2, \quad \hat{\mathbf{h}}_1 = \Omega b\tilde{\mathbf{e}}_1, \quad \mathbf{h}_k = \hat{\mathbf{h}}_k = 0 \quad (\ k \geq 2\ ). \tag{7.5}$$

The constants $\hat{\mathbf{g}}_k$ and $\mathbf{g}_k$ are then obtained from $(5.20)_1$ and $(5.20)_2$, respectively. Note that the angular rotation Ω is still unknown. To determine it, we consider the equilibrium of moment which gives (Chung, 1995; Ting and Yan, 1991)

$$\Omega = \frac{T}{\pi U}, \tag{7.6}$$

where

$$\begin{aligned} U &= b^2 H_{11}^{-1} + a^2 H_{22}^{-1} + 2ab(H^{-1}S)_{21}, \\ &= \bar{\mathbf{c}}^{\mathrm{T}}(\mathbf{H}^{-1} + i\mathbf{H}^{-1}\mathbf{S})\mathbf{c} = \bar{\mathbf{c}}^{\mathrm{T}}\mathbf{Z}\mathbf{c}. \end{aligned} \tag{7.7}$$

In the above $\mathbf{c}^{\mathrm{T}} = [-ib,\ a,\ 0,\ 0]$ and $\mathbf{Z}$ is the surface impedance tensor (Lothe and Barnett, 1976; Kuo and Barnett, 1991). Hence, one obtains

$$\hat{\mathbf{t}}_{\mathrm{m}} = \frac{1}{\rho(\psi)}\frac{\hat{\mathbf{f}}}{2\pi} + \frac{T}{\pi U}\mathbf{H}^{-1}\left[\frac{a}{b}\sin\omega\,\tilde{\mathbf{e}}_2 + \frac{b}{a}\cos\omega\,\tilde{\mathbf{e}}_1 - \mathbf{S}(\cos\omega\,\tilde{\mathbf{e}}_2 - \sin\omega\,\tilde{\mathbf{e}}_1)\right]. \tag{7.8}$$

For a circular rigid inclusion $\rho(\psi) = a = b = l$, (7.8) is simplified to

$$\hat{\mathbf{t}}_{\mathrm{m}} = \frac{1}{2\pi l}\hat{\mathbf{f}} + \frac{T}{\pi U}\mathbf{H}^{-1}[\cos\omega\,\tilde{\mathbf{e}}_1 + \sin\omega\,\tilde{\mathbf{e}}_2 + \mathbf{S}(\sin\omega\,\tilde{\mathbf{e}}_1 - \cos\omega\,\tilde{\mathbf{e}}_2)]. \tag{7.9}$$

In the case of zero torque, one obtains from (7.8)

$$\hat{\mathbf{t}}_{\mathrm{m}} = \frac{1}{\rho(\psi)}\frac{\hat{\mathbf{f}}}{2\pi}. \tag{7.10}$$

This means that $\rho(\psi)\hat{\mathbf{t}}_{\mathrm{m}}$ is a constant which is consistent with our observation in (6.15) and (6.23). Furthermore, for a circular rigid inclusion, (7.9) reduces to

$$\hat{\mathbf{t}}_{\mathrm{m}} = \begin{bmatrix} \mathbf{t}_{\mathrm{m}} \\ D_{\mathrm{m}} \end{bmatrix} = \frac{1}{2\pi l}\begin{bmatrix} \mathbf{f} \\ -\lambda \end{bmatrix}. \tag{7.11}$$

This means that the traction $\mathbf{t}_{\mathrm{m}}$ along the circular interface is a constant in the direction of $\mathbf{f}$. This phenomena was also observed in the purely elastic case (Ting and Yan, 1991).

To find the generalized hoop stress vector $\hat{\mathbf{t}}_{\mathrm{n}}$, (5.33) is employed in which $\hat{\mathbf{t}}_{\mathrm{m}}$ is given in (7.8) and (7.9) for elliptic and circular rigid inclusion of electric conductor, respectively.

8. A Piezoelectric Inclusion

In this section we consider an elliptic piezoelectric inclusion within a piezoelectric matrix. If the matrix is subjected to uniform fields at infinity, with either ε_{ij}^∞, E_i^∞ or σ_{ij}^∞, D_i^∞ prescribed such that $\varepsilon_{33}^\infty = 0$ and $E_3^\infty = 0$, then the uniform field solutions in the absence of an elliptic inclusion can be written as (Hwu and Ting, 1989)

$$\mathbf{u}^\infty = x_1 \boldsymbol{\varepsilon}_1^\infty + x_2 \boldsymbol{\varepsilon}_2^\infty, \quad \boldsymbol{\phi}^\infty = x_1 \mathbf{t}_2^\infty - x_2 \mathbf{t}_1^\infty, \tag{8.1}$$

in which

$$\boldsymbol{\varepsilon}_1^\infty = \mathbf{u}_{,1}^\infty = \begin{bmatrix} \varepsilon_{11}^\infty \\ 0 \\ 2\varepsilon_{13}^\infty \\ -E_1^\infty \end{bmatrix}, \quad \boldsymbol{\varepsilon}_2^\infty = \mathbf{u}_{,2}^\infty = \begin{bmatrix} 2\varepsilon_{21}^\infty \\ \varepsilon_{22}^\infty \\ 2\varepsilon_{23}^\infty \\ -E_2^\infty \end{bmatrix} \tag{8.2}$$

and

$$\mathbf{t}_1^\infty = \left[\sigma_{11}^\infty,\ \sigma_{12}^\infty,\ \sigma_{13}^\infty,\ D_1^\infty\right]^T = -\boldsymbol{\phi}_{,2}^\infty, \quad \mathbf{t}_2^\infty = \left[\sigma_{21}^\infty,\ \sigma_{22}^\infty,\ \sigma_{23}^\infty,\ D_2^\infty\right]^T = \boldsymbol{\phi}_{,1}^\infty. \tag{8.3}$$

Note that $(8.1)_1$ is unique up to a rigid body motion and a constant electrostatic potential. If ε_{ij}^∞, E_i^∞ are given, σ_{ij}^∞, D_i^∞ can be obtained by using (1.1), (1.4), and

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right). \tag{8.4}$$

If σ_{ij}^∞, D_i^∞ are known, the following constitutive equations are employed (Sosa, 1991):

$$\varepsilon_{ij} = S_{ijks}^D \sigma_{ks} + g_{kij} D_k, \quad E_i = -g_{iks}\sigma_{ks} + \beta_{ik}^\sigma D_k. \tag{8.5}$$

The zero element in $\boldsymbol{\varepsilon}_1^\infty$ implies that there is no rotation of the x_1 axis with respect to the x_3 axis. By means of superposition the field solutions in the piezoelectric matrix are (Hwu and Ting, 1989)

$$\left.\begin{aligned} \mathbf{u} &= \mathbf{u}^\infty + 2Re\left\{\mathbf{A} < \zeta_*^{-1} > \mathbf{A}^T\right\}\mathbf{g}_1 + 2Re\left\{\mathbf{A} < \zeta_*^{-1} > \mathbf{B}^T\right\}\mathbf{h}_1 \\ \boldsymbol{\phi} &= \boldsymbol{\phi}^\infty + 2Re\left\{\mathbf{B} < \zeta_*^{-1} > \mathbf{A}^T\right\}\mathbf{g}_1 + 2Re\left\{\mathbf{B} < \zeta_*^{-1} > \mathbf{B}^T\right\}\mathbf{h}_1 \end{aligned}\right\}. \tag{8.6}$$

The first terms of (8.6) arise from the uniform fields applied at infinity without an inclusion and are given in (8.1). The remaining terms are due to the presence of elliptic inclusion and are equivalent to (5.13) with $k = 1$. Thus, all the derivations obtained in previous sections can be employed. Along the elliptic inclusion boundary Γ, (8.6) is reduced to

$$\begin{aligned} \mathbf{u}|_\Gamma &= \mathbf{u}^\infty|_\Gamma + \cos\psi\,\mathbf{h}_1 - \sin\psi\,\hat{\mathbf{h}}_1 \\ &= a\cos\psi\,\boldsymbol{\varepsilon}_1^\infty + b\sin\psi\,\boldsymbol{\varepsilon}_2^\infty + \cos\psi\,\mathbf{h}_1 - \sin\psi\,\hat{\mathbf{h}}_1, \end{aligned} \tag{8.7}$$

$$\begin{aligned} \boldsymbol{\phi}|_\Gamma &= \boldsymbol{\phi}^\infty|_\Gamma + \cos\psi\,\mathbf{g}_1 - \sin\psi\,\hat{\mathbf{g}}_1 \\ &= a\cos\psi\,\mathbf{t}_2^\infty - b\sin\psi\,\mathbf{t}_1^\infty + \cos\psi\,\mathbf{g}_1 - \sin\psi\,\hat{\mathbf{g}}_1, \end{aligned} \tag{8.8}$$

in which (5.1), (5.17), and (8.1) are employed.

The solutions inside the piezoelectric inclusion are assumed uniform and have the form

$$\mathbf{u}^\mathrm{o} = x_1\boldsymbol{\varepsilon}_1^\mathrm{o} + x_2\boldsymbol{\varepsilon}_2^\mathrm{o}, \quad \boldsymbol{\phi}^\mathrm{o} = x_1\mathbf{t}_2^\mathrm{o} - x_2\mathbf{t}_1^\mathrm{o}, \tag{8.9}$$

where

$$\boldsymbol{\varepsilon}_1^\mathrm{o} = \mathbf{u}_{,1}^\mathrm{o} = \begin{bmatrix} \varepsilon_{11}^o \\ \Omega \\ 2\,\varepsilon_{13}^o \\ -E_1^o \end{bmatrix}, \quad \boldsymbol{\varepsilon}_2^\mathrm{o} = \mathbf{u}_{,2}^\mathrm{o} = \begin{bmatrix} 2\,\varepsilon_{21}^o - \Omega \\ \varepsilon_{22}^o \\ 2\,\varepsilon_{23}^o \\ -E_2^o \end{bmatrix}, \tag{8.10}$$

and

$$\mathbf{t}_1^\mathrm{o} = \left[\sigma_{11}^o,\ \sigma_{12}^o,\ \sigma_{13}^o,\ D_1^o\right]^\mathrm{T} = -\boldsymbol{\phi}_{,2}^\mathrm{o}, \quad \mathbf{t}_2^\mathrm{o} = \left[\sigma_{21}^o,\ \sigma_{22}^o,\ \sigma_{23}^o,\ D_2^o\right]^\mathrm{T} = \boldsymbol{\phi}_{,1}^\mathrm{o}. \tag{8.11}$$

The constant Ω in $\boldsymbol{\varepsilon}_1^\mathrm{o}$ and $\boldsymbol{\varepsilon}_2^\mathrm{o}$ represents the rotation of the x_1-axis in the inclusion. Along the elliptic inclusion boundary Γ, (8.9) and (5.1) give

$$\mathbf{u}^\mathrm{o}|_\Gamma = a\cos\psi\,\boldsymbol{\varepsilon}_1^\mathrm{o} + b\sin\psi\,\boldsymbol{\varepsilon}_2^\mathrm{o}, \quad \boldsymbol{\phi}^\mathrm{o}|_\Gamma = a\cos\psi\,\mathbf{t}_2^\mathrm{o} - b\sin\psi\,\mathbf{t}_1^\mathrm{o}. \tag{8.12}$$

The continuity condition states that

$$\mathbf{u}|_\Gamma = \mathbf{u}^\mathrm{o}|_\Gamma, \quad \boldsymbol{\phi}|_\Gamma = \boldsymbol{\phi}^\mathrm{o}|_\Gamma. \tag{8.13}$$

Employing (8.7), (8.8), and (8.12) in (8.13) leads to

$$\begin{aligned} a\boldsymbol{\varepsilon}_1^{\infty} + \mathbf{h}_1 &= a\boldsymbol{\varepsilon}_1^{o}, \quad b\boldsymbol{\varepsilon}_2^{\infty} - \hat{\mathbf{h}}_1 = b\boldsymbol{\varepsilon}_2^{o}, \\ a\mathbf{t}_2^{\infty} + \mathbf{g}_1 &= a\mathbf{t}_2^{o}, \quad b\mathbf{t}_1^{\infty} + \hat{\mathbf{g}}_1 = b\mathbf{t}_1^{o}. \end{aligned} \tag{8.14}$$

In order to solve $\mathbf{h}_1$, $\mathbf{g}_1$, $\boldsymbol{\varepsilon}_1^{o}$, $\boldsymbol{\varepsilon}_2^{o}$, $\mathbf{t}_1^{o}$, and $\mathbf{t}_2^{o}$, two more equations are required. Application of (3.15) to the matrix and the inclusion yields

$$\mathbf{N}\begin{bmatrix} \boldsymbol{\varepsilon}_1^{\infty} \\ \mathbf{t}_2^{\infty} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\varepsilon}_2^{\infty} \\ -\mathbf{t}_1^{\infty} \end{bmatrix}, \quad \mathbf{N}^{o}\begin{bmatrix} \boldsymbol{\varepsilon}_1^{o} \\ \mathbf{t}_2^{o} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\varepsilon}_2^{o} \\ -\mathbf{t}_1^{o} \end{bmatrix}. \tag{8.15}$$

With (5.18) we rewrite (8.14) in matrix form as

$$\begin{bmatrix} \mathbf{h}_1 \\ \mathbf{g}_1 \end{bmatrix} = a\begin{bmatrix} \boldsymbol{\varepsilon}_1^{o} \\ \mathbf{t}_2^{o} \end{bmatrix} - a\begin{bmatrix} \boldsymbol{\varepsilon}_1^{\infty} \\ \mathbf{t}_2^{\infty} \end{bmatrix}, \quad \begin{bmatrix} \mathbf{S} & \mathbf{H} \\ -\mathbf{L} & \mathbf{S}^{T} \end{bmatrix}\begin{bmatrix} \mathbf{h}_1 \\ \mathbf{g}_1 \end{bmatrix} = b\begin{bmatrix} \boldsymbol{\varepsilon}_2^{\infty} \\ -\mathbf{t}_1^{\infty} \end{bmatrix} - b\begin{bmatrix} \boldsymbol{\varepsilon}_2^{o} \\ -\mathbf{t}_1^{o} \end{bmatrix}. \tag{8.16}$$

Employing (3.13) and (8.15), $\boldsymbol{\varepsilon}_1^{o}$, $\mathbf{t}_2^{o}$ are obtained as

$$\left\{\frac{b}{a}\mathbf{N}^{0} + \begin{bmatrix} \mathbf{S} & \mathbf{H} \\ -\mathbf{L} & \mathbf{S}^{T} \end{bmatrix}\right\}\begin{bmatrix} \boldsymbol{\varepsilon}_1^{o} \\ \mathbf{t}_2^{o} \end{bmatrix} = \left\{\frac{b}{a}\mathbf{N} + \begin{bmatrix} \mathbf{S} & \mathbf{H} \\ -\mathbf{L} & \mathbf{S}^{T} \end{bmatrix}\right\}\begin{bmatrix} \boldsymbol{\varepsilon}_1^{\infty} \\ \mathbf{t}_2^{\infty} \end{bmatrix} \tag{8.17}$$

assuming that the matrix on the left is non-singular. The constants $\mathbf{h}_1$, $\mathbf{g}_1$ and $\boldsymbol{\varepsilon}_2^{o}$, $\mathbf{t}_1^{o}$ are then computed from $(8.16)_1$ and $(8.15)_2$, respectively. The rigid body rotation Ω can easily be determined from $(8.16)_1$ as

$$\Omega = \frac{(\mathbf{h}_1)_2}{a}. \tag{8.18}$$

The generalized stress vector along the elliptic inclusion boundary Γ is given by

$$\begin{aligned} \hat{\mathbf{t}}_m &= \boldsymbol{\phi}_{,n} = \frac{\partial\boldsymbol{\phi}|_{\Gamma}}{\rho(\psi)\partial\psi}, \\ &= \frac{-1}{\rho(\psi)}\left[a\sin\psi\,\mathbf{t}_2^{\infty} + b\cos\psi\,\mathbf{t}_1^{\infty}\right] + \frac{-1}{\rho(\psi)}\left[\mathbf{g}_1\sin\psi + \hat{\mathbf{g}}_1\cos\psi\right] \end{aligned} \tag{8.19}$$

when $(8.8)_2$ is used. Note that the second part of (8.19) has the same form as $(5.23)_2$ with $k = 1$. With $(8.8)_1$, the generalized hoop stress vector is

$$\hat{\mathbf{t}}_n = \boldsymbol{\phi}_{,m} = \boldsymbol{\phi}^{\infty}_{,m} + \left[\cos\psi\,\mathbf{g}_1 - \sin\psi\,\hat{\mathbf{g}}_1\right]_{,m}. \tag{8.20}$$

Observing that $\cos\psi\,\mathbf{g}_1 - \sin\psi\,\hat{\mathbf{g}}_1$ is equivalent to $\boldsymbol{\phi}^{\mathrm{II}}|_{\Gamma}$ given in $(5.17)_2$ with $k = 1$, its derivative with respect to m can be computed as (Chung, 1995; Hwu and Ting, 1989)

$$\begin{aligned}\left[\cos\psi\,\mathbf{g}_1 - \sin\psi\,\hat{\mathbf{g}}_1\right]_{,m} = \boldsymbol{\phi}^{\mathrm{II}}_{,m} &= \mathbf{N}_3(\omega)\,\mathbf{u}^{\mathrm{II}}_{,n} + \mathbf{N}_1^{\mathrm{T}}(\omega)\,\boldsymbol{\phi}^{\mathrm{II}}_{,n},\\ &= \mathbf{N}_3(\omega)\frac{\partial \mathbf{u}^{\mathrm{II}}|_{\Gamma}}{\rho(\psi)\,\partial\psi} + \mathbf{N}_1^{\mathrm{T}}(\omega)\frac{\partial \boldsymbol{\phi}^{\mathrm{II}}|_{\Gamma}}{\rho(\psi)\,\partial\psi}.\end{aligned} \tag{8.21}$$

With $\mathbf{u}^{\mathrm{II}}|_{\Gamma}$ given by $(5.17)_1$, when $k = 1$, we have

$$\hat{\mathbf{t}}_n = \boldsymbol{\phi}^{\infty}_{,m} - \left\{\frac{\mathbf{N}_3(\omega)}{\rho(\psi)}\left[\mathbf{h}_1\sin\psi + \hat{\mathbf{h}}_1\cos\psi\right] + \frac{\mathbf{N}_1^{\mathrm{T}}(\omega)}{\rho(\psi)}\left[\mathbf{g}_1\sin\psi + \hat{\mathbf{g}}_1\cos\psi\right]\right\}. \tag{8.22}$$

Again, the second part of (8.22) is readily obtained by observing that the result stated in $(5.28)_2$ can be employed with $k = 1$. From (5.4), (5.25), and $(8.1)_2$, $\boldsymbol{\phi}^{\infty}_{,m}$ is found to be

$$\boldsymbol{\phi}^{\infty}_{,m} = \boldsymbol{\phi}^{\infty}_{,1}(-\sin\omega) + \boldsymbol{\phi}^{\infty}_{,2}(\cos\omega) = -\mathbf{t}_2^{\infty}\frac{b}{\rho(\phi)}\cos\psi + \mathbf{t}_1^{\infty}\frac{a}{\rho(\psi)}\sin\psi. \tag{8.23}$$

Combining (8.22) and (8.23) yields, with (5.18),

$$\begin{aligned}\hat{\mathbf{t}}_n = &\frac{-1}{\rho(\psi)}\left[\mathbf{t}_2^{\infty}b\cos\psi - \mathbf{t}_1^{\infty}a\sin\psi\right]\\ &- \frac{1}{\rho(\psi)}\left\{\mathbf{N}_3(\omega)\left[\mathbf{h}_1\sin\psi + \left(\mathbf{S}\mathbf{h}_1 + \mathbf{H}\mathbf{g}_1\right)\cos\psi\right]\right\}\\ &- \frac{1}{\rho(\psi)}\left\{\mathbf{N}_1^{\mathrm{T}}(\omega)\left[\mathbf{g}_1\sin\psi + \left(-\mathbf{L}\mathbf{h}_1 + \mathbf{S}^{\mathrm{T}}\mathbf{g}_1\right)\cos\psi\right]\right\}.\end{aligned} \tag{8.24}$$

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References

Barnett, D.M. and Lothe, J. (1975), Dislocations and line charges in anisotropic piezoelectric

insulators. *Phys. Status Solidi B* **67**, pp. 105-111.

Bary, N.K. (1964), *A Treatise on Trigonometric Series* **V2**, pp. 51-52. Macmillan, New York.

Chung, M.Y. (1995), Two-dimensional problems of piezoelectric solids with an elliptic inclusion. Ph.D. Dissertation, University of Illinois at Chicago.

Eshelby, J.D., Read, W.T., and Shockley W. (1953), Anisotropic elasticity with applications to dislocation theory. *Acta Metallurgica* **1**, pp. 251-259.

Hwu, Chyanbin and Ting, T.C.T. (1989), Two-dimensional problems of the anisotropic elastic solids with an elliptic inclusion. *Q. J. Mech. Appl. Math.* **42**, pp. 553-572.

Hwu, Chyanbin and Ting, T.C.T. (1990), Solutions for the anisotropic elastic wedges at critical wedge angles. *J. Elasticity* **24**, pp. 1-20.

Kuo, C.M. and Barnett, D.M. (1991), Stress Singularities of interface cracks in bonded piezoelectric half-spaces. In *Modern Theory of Anisotropic Elasticity and Applications*, pp. 33-51. (Edited by J.J. Wu, T.C.T. Ting and D.M. Barnett). SIAM Proceedings Series, SIAM, Philadelphia.

Lothe, J and Barnett, D.M. (1976), Integral formalism for surface waves in piezoelectric crystals. Existence considerations. *J. Appl. Phys.* **47**, pp. 1799-1807.

Lothe, J. and Barnett, D.M. (1977), Further development of the theory for surfaces waves in piezoelectric crystals. *Physica Norvegica* **V8**, pp. 239-254.

Pak, Y. Eugene (1992), Circular inclusion problem in antiplane piezoelectricity. *Int. J. Solids Structures* **29**, pp. 2403-2419.

Sosa, Horacio (1991), Plane problems in piezoelectric media with defects. *Int. J. Solids Structures* **28**, pp. 491-505.

Stroh, A.N. (1958), Dislocations and cracks in anisotropic elasticity. *Phil. Mag.* **3**, pp. 625-646.

Stroh, A.N. (1962), Steady state problems in anisotropic elasticity. *J. Math. Phys.* **41**, pp. 77-103.

Suo, Z., Kuo, C.M., Barnett, D.M., and Willis, J.R. (1992), Fracture mechanics for piezoelectric ceramics. *J. Mech. Phys. Solids* **40**, pp. 739-765.

Tiersten, H.F. (1969), *Linear Piezoelectric Plate Vibrations*. Plenum Press, New York.

Ting, T.C.T. (1986), Explicit solution and invariance of the singularities at an interface crack in anisotropic composites. *Int. J. Solids Structures* **22**, pp. 965-983.

Ting, T.C.T. (1988a), Line forces and dislocations in anisotropic elastic composite wedges and spaces. *Phys. Status Solidi B* **146**, pp. 81-90.

Ting, T.C.T. (1988b), Some identities and the structure of N_i in the Stroh formalism of anisotropic elasticity. *Q. Appl. Math.* **46**, pp. 109-120.

Ting, T.C.T. and Yan, Gongpu (1991), The anisotropic elastic solid with an elliptic hole or rigid inclusion. *Int. J. Solids Structures* **27**, pp. 1879-1894.

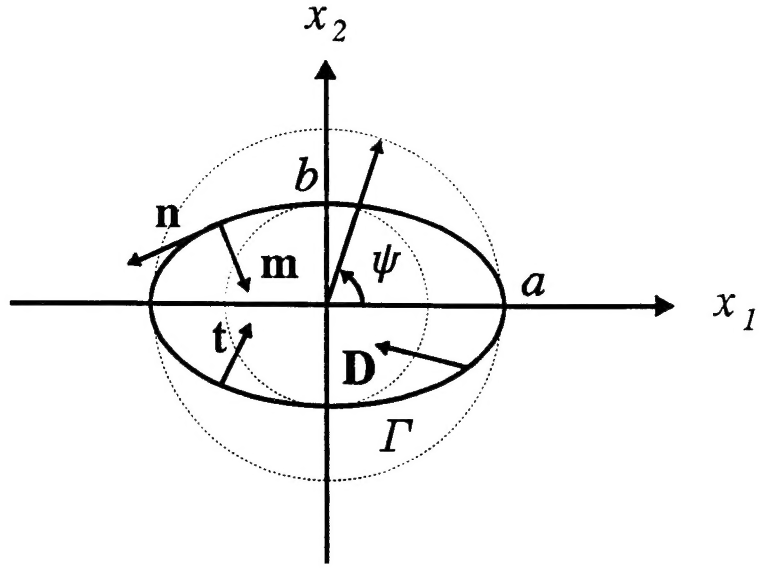

Fig.1 An ellipse in the (x_1, x_2) plane

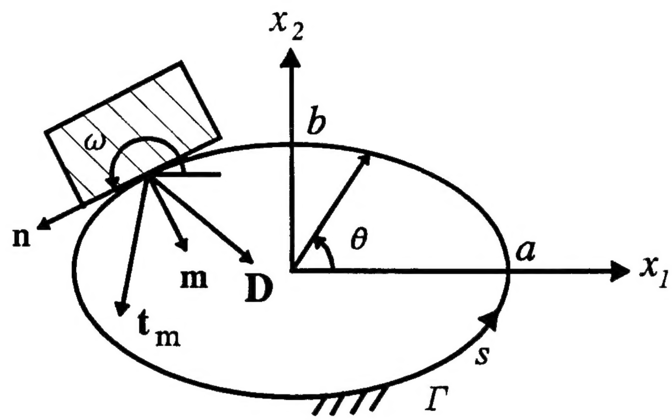

Fig.2 Generalized stress vector along the elliptic boundary

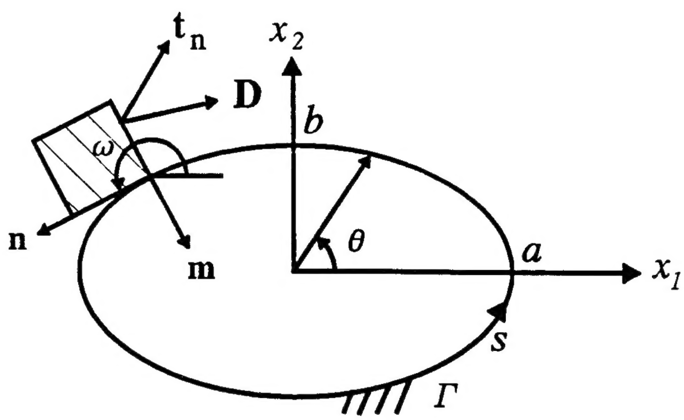

Fig.3 Generalized hoop stress vector along the elliptic boundary